

CH WITH NO OSTASZEWSKI SPACES

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ABSTRACT. There are models of CH without Ostaszewski spaces. If X is locally compact and sub-Ostaszewski, there is a forcing P_X which does not add reals and which forces “ X is not sub-Ostaszewski”.

1. INTRODUCTION

Ostaszewski spaces, originally constructed under \diamond to answer a question about perfect normality (see [4]), have since proved useful both as counterexamples and as paradigms in many areas of general topology, and thus they are of some interest in their own right. As is often the case, the reflection argument implicit in a \diamond construction can be easily modified to a forcing construction under, e.g., the partial order that simultaneously adds uncountably many Cohen reals, and variations with stronger properties were constructed using these methods. But it was not known if reflection was needed — in particular, it was unknown if Ostaszewski spaces could be constructed from CH alone. The aim of this paper is to show that they cannot.

Theorem 1. *There are models of $\text{ZFC} + \text{CH}$ in which there are no Ostaszewski spaces.*

The consistency is relative to the consistency of ZFC; our results require no large cardinals. Along the way, we establish the following:

Theorem 2. *Suppose X is a locally compact sub-Ostaszewski space. Then there is a notion of forcing P_X that adds no reals such that*

$$(1.1) \quad \Vdash_{P_X} \text{“} X \text{ is not a sub-Ostaszewski space”}.$$

Our proof defines P_X so that it adds an uncountable set, each of whose countable subsets has compact closure. To show that P_X is proper and adds no reals, we need only the assumptions of Theorem 2. In order to show that such forcings can be iterated without adding reals, we use the full definition of Ostaszewski spaces.

2. TOPOLOGICAL BACKGROUND

We gather here several elementary results concerning sub-Ostaszewski spaces; all are folklore.

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Definition 2.1. A Hausdorff space X is a sub-Ostaszewski space if X is uncountable and every closed set is either countable or co-countable. If in addition X is regular, countably compact, and non-compact, we call X an Ostaszewski space.

Proposition 2.2. *A sub-Ostaszewski space has at most one point with no countable neighborhood.*

Proof. This follows immediately because in a sub-Ostaszewski space uncountable open sets cannot be disjoint, and thus given two points, the fact that our space is Hausdorff tells us that at least one of the points has a countable neighborhood. \square

Proposition 2.3. *A locally countable space is hereditarily separable if and only if it has no uncountable discrete subspace.*

Proof. Clearly an uncountable discrete subspace is not separable, so this proves one direction. For the other direction, suppose Y is an uncountable non-separable subspace of a locally countable space X . Then it is straightforward to construct a subset $Z = \{x_\alpha : \alpha < \omega_1\}$ of Y and a sequence $\{U_\alpha : \alpha < \omega_1\}$ of countable open subsets of X such that $x_\alpha \in U_\alpha$ and $U_\alpha \cap \{x_\beta : \beta < \alpha\} = \emptyset$. Then Z is an uncountable discrete subspace of X . \square

Proposition 2.4. *A sub-Ostaszewski space is hereditarily separable.*

Proof. Let X be a sub-Ostaszewski space. By removing a point if necessary, we can assume that X is locally countable. If Y is an uncountable discrete subspace of X and we partition Y into uncountable sets Y_0 and Y_1 , then $\text{cl } Y_0 \cap Y_1 = \emptyset$, so $\text{cl } Y_0$ is uncountable but not co-countable, a contradiction. \square

Proposition 2.5. *Locally countable regular spaces are 0-dimensional.*

Proof. Given X locally countable, let U be a countable neighborhood of some point $x \in X$. Since we assume X is regular, there is an open neighborhood V of x such that $\text{cl } V \subseteq U$. If $U = \text{cl } V$ we are done. Otherwise, we use the fact that countable spaces are completely regular to get a continuous map $f : \text{cl } V \rightarrow [0, 1]$ which sends x to 0 and $\text{cl } V \setminus V$ to 1. Let \hat{f} be a continuous extension of f to X that sends $X \setminus V$ to 1. Since $\text{ran } \hat{f}$ is countable, we can find an $r \in [0, 1] \setminus \text{ran } \hat{f}$, and then $\hat{f}^{-1}([0, r])$ is a clopen neighborhood of x contained in U . \square

Proposition 2.6. *Ostaszewski spaces are locally countable.*

Proof. Let X be an Ostaszewski space and suppose x has no countable neighborhood. Then every open neighborhood of x is co-countable, which by countable compactness means every neighborhood of x is co-compact. Hence X is compact. \square

Proposition 2.7. *Locally countable 0-dimensional countably compact spaces are locally compact; hence Ostaszewski spaces are locally compact.*

Proof. This is immediate because a countable clopen set in a countably compact space is compact. \square

Proposition 2.8. *A locally countable locally compact space is scattered.*

Proof. If Y is a subset of such a space, we can pick some countable open set U that meets Y and has compact closure. By a theorem of Mazurkiewicz and Sierpinski, countable compact spaces are homeomorphic to ordinals, so there is some isolated point x in $U \cap Y$. \square

Proposition 2.9. *A locally countable scattered space of size ω_1 is right-separated in type ω_1 , i.e., the space can be well-ordered in type ω_1 so that initial segments are open.*

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be locally countable and scattered. We re-enumerate X as $\{y_\alpha : \alpha < \omega_1\}$ where y_0 is isolated in X , and each y_α is x_β where β is the least such that x_β is isolated in $X \setminus \{y_\gamma : \gamma < \alpha\}$. \square

Proposition 2.10. *A sub-Ostaszewski space has size ω_1 .*

Proof. Let X be a sub-Ostaszewski space. If $|X| \geq \omega_2$, then by the free set lemma for countable sets, there is a collection \mathcal{U} of countable open sets, $|\mathcal{U}| = \omega_2$, so that for $U \in \mathcal{U}$, $U \setminus \bigcup[\mathcal{U} \setminus \{U\}] \neq \emptyset$. But then X has a discrete subspace of size ω_2 , contradicting Proposition 2.4. \square

From now on, we assume X is a topological space with ω_1 as an underlying set and initial segments open — the above propositions show that in order to destroy all Ostaszewski spaces, it is sufficient to consider only those spaces X with this form. Our convention is that topological notions refer to the topology τ on X , while ordering notions (e.g. “cofinal”, “initial segment”) refer to the usual order on ω_1 .

We now introduce some definitions and facts that are needed for our proof.

Definition 2.11. We write $\delta = X_\delta$ if every open set meeting $X \setminus \delta$ contains a sequence cofinal in δ .

This choice of notation comes from the notation for the Cantor-Bendixson decomposition of scattered spaces.

Proposition 2.12. *Let X be hereditarily separable and N be a countable elementary submodel of $H(\lambda)$ with $X \in N$. If $\delta = N \cap \omega_1$, then $\delta = X_\delta$.*

Proof. Let U be open in X with $U \setminus \delta \neq \emptyset$. For purposes of contradiction, assume that $U \cap \delta$ is bounded by some $\gamma < \delta$. Since $\gamma \in N$, we can find (using hereditary separability) a countable set $D \subseteq X \setminus \gamma$ in N such that $\text{cl } D = X \setminus \gamma$. In particular, $U \cap D$ cannot be empty, and this contradicts our choice of γ . \square

An immediate corollary of the above proposition is that the set $\{\delta : \delta = X_\delta\}$ is closed and unbounded in ω_1 if X is hereditarily separable.

Definition 2.13. We say a set Y is cofinally convergent in δ if

1. $Y \subseteq \delta$ is unbounded in δ ,
2. $\text{cl } Y$ is compact,
3. there is some point $\text{Top}(Y)$ such that every increasing ω -sequence in Y cofinal in δ converges to $\text{Top}(Y)$.

We note that if Y is cofinally convergent in δ , then $\text{Top}(Y)$ is unique and $\text{Top}(Y) \geq \delta$.

Proposition 2.14. *Let Y be cofinal in δ where $\delta = X_\delta$. If X is locally compact and Y is open with compact closure, then there is some open $Z \subseteq Y$ cofinally convergent in δ .*

Proof. Let $\theta \geq \delta$ be the minimum element of $\text{cl } Y \setminus \delta$. If U is a compact open neighborhood of θ with $\sup U = \theta$, then $Y \cap U$ is the desired set. \square

Corollary 2.15. *If X is countably compact and locally compact, $Y \subseteq X$ and Y is open and cofinal in $\delta = X_\delta$, then Y has an open subset Z cofinally convergent in δ .*

Proposition 2.16. *If Y is open and cofinally convergent in δ where $\delta = X_\delta$, then for all compact open sets U , either $Y \setminus U$ is open and bounded below δ , or $Y \cap U$ is open and bounded below δ ; $Y \setminus U$ is bounded below δ if and only if $\text{Top}(Y) \in U$.*

3. TOTAL PROPERNESS AND ITERATED FORCING

We assume familiarity with Shelah's work on proper forcing as presented in [5], [6], or [3]; in particular, we assume the reader is used to working with countable elementary submodels of $H(\lambda)$ — the sets whose hereditary cardinality is less than λ — where λ is some large enough regular cardinal. We assume (without loss of generality) that all of the notions of forcing with which we work are separative, i.e., if q is not an extension of p , then p has an extension r that is incompatible with q . Notice that if P is separative and both p and q are in P , then $p \Vdash q \in \dot{G}_P$ if and only if p extends q ; this will be of use later.

Our conventions regarding iterated forcing are standard (see [3] for example). If $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ is a countable support iteration, we adopt the convention that P_0 is the trivial one element partial order for notational convenience. We will write \Vdash_α instead of \Vdash_{P_α} . We will also make use of standard facts concerning quotient forcing. For example, if $\alpha < \beta < \kappa$, then by P_β/\dot{G}_α , we mean a P_α name such that

$$(3.1) \quad \Vdash_\alpha P_\beta/\dot{G}_\alpha = \{p \in P_\beta : p \restriction \alpha \in \dot{G}_\alpha\}$$

where \dot{G}_α is the canonical P_α name for the generic set adjoined by P_α . As is well-known, forcing with P_β is essentially the same as forcing with $P_\alpha * P_\beta/\dot{G}_\alpha$. We use, at some point, the fact that if $r \in P_\alpha$, \dot{s} is a P -name forced by r to lie in P_β/\dot{G}_α , and there is a countable set of ordinals S in V so that r forces that the support of \dot{s} is a subset of S , then $r * \dot{s} \in P_\alpha * P_\beta/\dot{G}_\alpha$ can be canonically translated into condition $r^+ \in P_\beta$ satisfying $r^+ \restriction \alpha = r$. The reader seeking more details should consult Section 4 of [3].

We now turn to the study of proper notions of forcing that add no new reals to the ground model. The notation $\text{Gen}(N, P)$ and $\text{Gen}^+(N, P)$ has its origins in [5].

Definition 3.1. Let P be a notion of forcing, let $N \prec H(\lambda)$ be countable with $P \in N$, and let $p \in N \cap P$. We define

1. $\text{Gen}(N, P) = \{G \subseteq N \cap P : G \text{ is an } N\text{-generic filter on } N \cap P\}$;
2. $\text{Gen}^+(N, P) = \{G \in \text{Gen}(N, P) : G \text{ has a lower bound in } P\}$;
3. $\text{Gen}(N, P, p) = \{G \in \text{Gen}(N, P) : p \in G\}$;
4. $\text{Gen}^+(N, P, p) = \text{Gen}(N, P, p) \cap \text{Gen}^+(N, P)$.

In the sequel, we will have occasion to speak of several objects that are generic in certain senses. In general, when we are speaking of a notion of forcing P , we let \dot{G} be the canonical P -name for the generic subset of P . When we are speaking of elements of some $\text{Gen}(N, P)$, we will use an overset bar to indicate that we are referring to an object that is generic only over the model N , and when the bar is lacking we are referring to honest-to-goodness generic subsets of P . Thus we will see phrases like “let $\bar{G} \in \text{Gen}(N, P)$ ” or “let $G \subseteq P$ be generic”.

Definition 3.2. With N and P as in the previous definition, we say a condition $q \in P$ is totally (N, P) -generic if q is a lower bound of some $\bar{G} \in \text{Gen}(N, P)$. We say

P is totally proper if for any countable $N \prec H(\lambda)$ containing P and any $p \in N \cap P$, there is a totally (N, P) -generic condition $q \leq p$.

We will invest a bit of time developing some of the basic properties of totally proper forcing. None of the proofs are very difficult.

Proposition 3.3. *For a notion of forcing P , the following conditions are equivalent:*

1. P is totally proper.
2. P is proper, and forcing with P adds no new function from ω to V .
3. P is proper, and forcing with P adds no new reals.

Proof. (1) implies (2): Suppose \dot{f} is a P -name for a function from ω to V , and let $p \in P$ be arbitrary. We find a $q \leq p$ and $f \in V$ such that

$$q \Vdash \dot{f} = f.$$

Choose a countable $N \prec H(\lambda)$ with $\{p, P, \dot{f}\} \in N$. For each n , let

$$(3.2) \quad D_n = \{r \in P : \text{for some } x \in V, r \Vdash \dot{f}(n) = x\}.$$

Note that D_n is dense open in P as the range of \dot{f} consists of ground model elements, and clearly $D_n \in N$.

Let $q \leq p$ be totally (N, P) -generic. For each n , choose $r_n \in N \cap D_n$ with $q \leq r_n$. Let $x_n \in V$ be such that $r_n \Vdash \dot{f}(n) = x_n$. If we define f (in V) by $f(n) = x_n$, then $q \Vdash \dot{f} = f$ as desired.

The implication (2) \rightarrow (3) is trivial, so we show next that (3) implies (1).

In regard to this, let relevant N and p be given, and let $\{D_n : n \in \omega\}$ enumerate the dense open subsets of P that are in N . Fix a function $g : \omega \times \omega \rightarrow P$ so that

$$(3.3) \quad D_n \cap N = \{g(m, n) : m \in \omega\}$$

and let $r \leq p$ be (N, P) -generic. This means that for all dense open $D_n \in N$, D_n is predense below r ; that is, for all $s \leq r$ there exists $t \in D_n \cap N$ such that s and t are compatible. Since $r \Vdash N \cap D_n \cap \dot{G}_P \neq \emptyset$, there is a P -name \dot{f} such that

$$(3.4) \quad r \Vdash \forall n [\dot{f}(n) \text{ is the least } m \text{ such that } g(m, n) \in \dot{G}_P].$$

Since forcing with P adds no new reals, there is a function $f \in \omega^\omega$ (in V) and $q \leq r$ such that $q \Vdash \dot{f} = f$. Now since q is an extension of r , we know that for each n , q forces that $g(f(n), n)$ is in \dot{G}_P . Since P is separative, this means that q actually extends $g(f(n), n)$, and so q is totally (N, P) -generic. Note that it is important that $g(f(n), n)$ is an honest-to-goodness member of P , and not just a P -name like $g(\dot{f}(n), n)$. \square

We remark that the notion of forcing, shooting a closed unbounded set through a stationary, co-stationary $S \subseteq \omega_1$ using countable conditions is an example of a notion of forcing that adds no new functions from ω to V that fails to be proper, and that Namba forcing (which changes the cofinality of \aleph_2 to \aleph_0) is an example of a notion of forcing that adds no new reals and yet adds a new function from ω to V .

Given a countable $N \prec H(\lambda)$ containing a notion of forcing P and a \bar{G} in $\text{Gen}(N, P)$, we may ask what happens if we interpret the P -names that are in N using \bar{G} as an oracle. In general, \bar{G} will not have enough information to give us a complete interpretation of $\dot{\tau}$. For example, if \dot{G} is the canonical P -name for the

generic object, then $\dot{G}[\bar{G}]$ is just \bar{G} — a truncated picture of what \dot{G} will look like in a real generic extension. On the other hand, if we look at $\dot{\omega}_1[\bar{G}]$ (where $\dot{\omega}_1$ is the canonical name for ω_1), we see that $\dot{\omega}_1[\bar{G}] = \omega_1$ — no truncation at all. The problem here is that N is not transitive; things are much nicer if we look at the Mostowski collapse of N and \bar{G} .

Definition 3.4. Let P be a notion of forcing, and let N be a countable elementary submodel of $H(\lambda)$ that contains P .

1. If G is a generic subset of P , then in $V[G]$ let

$$N[G] = \{\dot{\sigma}[G] : \dot{\sigma} \text{ a } P\text{-name in } N\}.$$

2. $N[\dot{G}]$ is a P -name forced by every condition to be the set $N[G]$ defined above.
3. If \bar{G} is in $\text{Gen}(N, P)$ and $\dot{\sigma}$ is a P -name from N , then we define $\dot{\sigma}[\bar{G}]$ by setting

$$\dot{\sigma}[\bar{G}] = \pi(\dot{\sigma})[\pi(\bar{G})]$$

where π is the Mostowski map taking $\mathcal{P}(N)$ to its transitive collapse.

4. If \bar{G} is in $\text{Gen}(N, P)$, then let

$$N[\bar{G}] = \{\dot{\sigma}[\bar{G}] : \dot{\sigma} \text{ a } P\text{-name in } N\}.$$

The first thing one should note is that $N[\bar{G}]$ is in the ground model, whereas $N[G]$ exists only in a generic extension. Assume now that our notion of forcing is totally proper, $\bar{G} \in \text{Gen}^+(N, P)$, and G is a generic subset of P containing a lower bound q for \bar{G} . Since q is totally (N, P) -generic, we know that $N[G]$ is a countable elementary submodel of $H(\lambda)[G]$ and $N \cap G = \bar{G}$ (see Lemma 3.6 below). In $V[G]$, if we take the transitive collapse of $N[G]$, we get $N[\bar{G}]$. Thus even though we need all of G to get $N[G]$, the collapse of $N[G]$ depends only on \bar{G} , and any lower bound q for \bar{G} forces that $N[\bar{G}]$ (in the ground model) is the transitive collapse of $N[\dot{G}]$ (in the extension).

Lemma 3.5. Suppose P is totally proper, N is a countable elementary submodel of some $H(\lambda)$, $p \in N$, and $\bar{G} \in \text{Gen}^+(N, P)$ with lower bound r . Then for a formula $\psi(x_0, \dots, x_n)$ and P -names $\dot{\sigma}_0, \dots, \dot{\sigma}_n$ from N , the following are equivalent:

1. $N[\bar{G}] \models \psi(\dot{\sigma}_0[\bar{G}], \dots, \dot{\sigma}_n[\bar{G}])$,
2. $r \Vdash N[\dot{G}] \models \psi(\dot{\sigma}_0, \dots, \dot{\sigma}_n)$,
3. $r \Vdash \psi(\dot{\sigma}_0, \dots, \dot{\sigma}_n)$.

Proof. Clearly (2) if and only if (3) because r forces that $N[\dot{G}]$ is an elementary submodel of $H(\lambda)[\dot{G}]$. For (1) if and only if (2), let G be any generic subset of P that contains r . In $V[G]$, the Mostowski collapse of $N[G]$ onto $N[\bar{G}]$ is an isomorphism of models, and by definition, $\pi(\dot{\sigma}_i[G]) = \dot{\sigma}_i[\bar{G}]$. \square

Notice in the above lemma that statement 1 does not depend on the lower bound r for \bar{G} . Thus if we verify that $N[\bar{G}] \models \psi$, then we know every lower bound for \bar{G} forces ψ to be true in the extension.

Lemma 3.6. If $\bar{G} \in \text{Gen}(N, P)$, then \bar{G} is a maximal filter on $N \cap P$.

Proof. This is immediate, because for each $p \in N \cap P$, the set of q that either extends p or is incompatible with p is dense in P and a member of N , hence \bar{G} contains such a q . \square

Note that as a corollary to the above lemma, if $\bar{G} \in \text{Gen}^+(N, P)$ and r is a lower bound for \bar{G} , then $r \Vdash N \cap \dot{G}_P = \bar{G}$. Even though there are lots of dense subsets of P that are not in N , $\dot{G}_P \cap N$ cannot grow any larger than \bar{G} . The above lemma is just a “local version” of the well-known fact that generic filters on a partial order must be maximal.

Lemma 3.7. *If P is totally proper and $G \subseteq P$ is generic, then G is downward σ -directed, i.e., given $\{p_n : n \in \omega\} \subseteq G$, there is a $p \in G$ such that $\forall n [p \leq p_n]$.*

Proof. In $V[G]$, let $\{p_n : n \in \omega\}$ be given. By Proposition 3.3, we know that $\{p_n : n \in \omega\} \in V$. Let D be the set of $q \in p$ that is either a lower bound for $\{p_n : n \in \omega\}$ or incompatible with some p_n . Since P is separative, this set is dense open in P . Since $D \in V$, there is some $q \in D \cap G$. Now the fact that G is a filter means that q must be a lower bound for $\{p_n : n \in \omega\}$. \square

Lemma 3.8. *Let P be totally proper, and let $N \in M$ be countable elementary submodels of $H(\lambda)$ with $P \in N$. If r is both (N, P) -generic and (M, P) -generic, then $r \Vdash N \cap \dot{G} \in \text{Gen}^+(N, P) \cap M$.*

Proof. Let G be any generic subset of P that contains r . Since r is (N, P) -generic, we know $N \cap G \in \text{Gen}(N, P)$, and by the downward σ -directedness of G we can find a lower bound for $N \cap G$. Thus we need only verify that $N \cap G$ is a member of M . To see this, define

$$(3.5) \quad D = \{p \in P : p \text{ is totally } (N, P)\text{-generic,} \\ \text{or } p \text{ has no totally } (N, P)\text{-generic extension}\}.$$

The set D is a member of M and is dense and open in P . Since $r \in G$ is (M, P) -generic, we know that there is some q in $M \cap D \cap G$. By the directedness of G , q has a totally (N, P) -generic extension and so q must be (N, P) -generic. By Lemma 3.6, $N \cap G = \{p \in N \cap P : q \leq p\}$, and so $N \cap G$ is definable from parameters in M . \square

One other strengthening of properness is relevant to our proof, so we need a few more definitions.

Definition 3.9. We say $\mathfrak{N} = \langle N_\xi : \xi \leq \alpha \rangle$ is an α -tower if for some large enough regular cardinal λ , each N_ξ is a countable elementary submodel of $H(\lambda)$, the sequence \mathfrak{N} is increasing and continuous at limits, and $\langle N_\zeta : \zeta \leq \xi \rangle \in N_{\xi+1}$. A notion of forcing P is said to be α -proper if for every α -tower \mathfrak{N} such that $P \in N_0$, and for every $p \in N_0 \cap P$, there is a $q \leq p$ that is (N_ξ, P) -generic for each $\xi \leq \alpha$. We say P is $< \omega_1$ -proper if P is α -proper for each $\alpha < \omega_1$.

Iterations of $< \omega_1$ -proper notions of forcing are dealt with in Chapter V of [5]; the same basic iteration theorems that are true for proper forcing, hold in the $< \omega_1$ -proper case as well, and we make use of this in the sequel.

4. ONE SPACE AT A TIME

Let X be a locally compact sub-Ostaszewski space. We may assume that ω_1 is the underlying set of X and that each initial segment of ω_1 is open in X . The forcing notion defined below is based on work of Abraham and Todorćević [1] as presented in [2].

Definition 4.1. Our notion of forcing P_X consists of pairs $p = (x_p, \Phi_p)$ where

1. $x_p : \delta \rightarrow 2$ for some $\delta < \omega_1$;
2. $[p] = x_p^{-1}(\{1\})$ has compact closure in X ;
3. Φ_p is a countable set of functions f such that
 - (a) $\text{dom}(f)$ is an uncountable subset of ω_1 ,
 - (b) for $\alpha \in \text{dom}(f)$, $f(\alpha)$ is a compact open neighborhood of α and $f(\alpha) \subseteq \alpha + 1$.

Such a function f is called a promise. A condition q extends a condition p (written $q \leq p$) if

4. $x_q \supseteq x_p$;
5. $\Phi_q \supseteq \Phi_p$;
6. for every $f \in \Phi_p$, the set $Y(f, p, q) = \{\xi \in \text{dom}(f) : [q] \setminus [p] \subseteq f(\xi)\}$ is uncountable, and $f \upharpoonright Y(f, p, q) \in \Phi_q$.

A condition p should be viewed as an approximation of a subset X' of X (this is $[p]$) together with some restraints on how this approximation can be extended (this is Φ_p). An $f \in \Phi_p$ promises that when we add new points to $[p]$, they will be contained in $f(\alpha)$ for uncountably many $\alpha \in \text{dom}(f)$. If we can show that this set X' adjoined by P_X is uncountable in the extension, then X is no longer hereditarily separable because each countable subset of X' will have compact (hence countable) closure by condition 2 above. Since sub-Ostaszewski spaces are hereditarily separable, we know that X will no longer be a sub-Ostaszewski space in the extension.

Lemma 4.2. *For every $p \in P_X$ and $\theta < \omega_1$, there is an extension q of p with $[q] \setminus \theta \neq \emptyset$.*

Proof. Assume this is not the case, and fix $p \in P_X$ and $\theta < \omega_1$ such that every extension q of p has $[q] \subseteq \theta + 1$. Now if $\alpha > \theta$, the only obstacle to adding α to $[p]$ is that there is some promise $f_\alpha \in \Phi_p$ such that $\{\xi \in \text{dom}(f_\alpha) : \alpha \in f_\alpha(\xi)\}$ is countable. Since Φ_p is countable, there is an uncountable set $Z \subseteq \omega_1 \setminus \theta$ and a single $f \in \Phi_p$ such that

$$(4.1) \quad \forall \alpha \in Z \ \{\xi \in \text{dom}(f) : \alpha \in f(\xi)\} \text{ is countable.}$$

Let $Y_0 = \text{dom}(f) \cap \text{cl } Z$. Since $\text{cl } Z$ is co-countable, we know that Y_0 is uncountable. Since X is hereditarily separable, there is a countable subset Z_0 of Z such that $\text{cl } Z_0 = \text{cl } Z$. Given $\beta \in Y_0$, there is some $z_\beta \in Z_0$ with $z_\beta \in f(\beta)$. Since Y_0 is uncountable and Z_0 is countable, there is a $z \in Z_0$ such that $\{\beta \in Y_0 : z \in f(\beta)\}$ is uncountable. This is a contradiction of (4.1) as $z \in Z$ and $Y_0 \subseteq \text{dom}(f)$. \square

Lemma 4.2 tells us that the set adjoined by P_X is cofinal in ω_1^V , but we have yet to show that ω_1^V is still uncountable in the extension. The remainder of this section will establish that P_X is totally proper, and hence ω_1 is preserved. To start, we need a definition.

Definition 4.3. Given a condition $p \in P_X$ and a dense open subset D of P_X , define

$$(4.2) \quad \text{Bad}(p, D) = \{\alpha \in \omega_1 \setminus \text{dom}(x_p) : \text{there is a compact open } U_\alpha \text{ with } \alpha \in U_\alpha \subseteq \alpha + 1 \text{ such that whenever } q \leq p \text{ and } q \in D, [q] \setminus [p] \not\subseteq U_\alpha\}.$$

Lemma 4.4. *For $p \in P_X$ and $D \subseteq P_X$ dense open, $\text{Bad}(p, D)$ is countable.*

Proof. Suppose this is not the case. Define a function f with domain $\text{Bad}(p, D)$ and for $\alpha \in \text{dom}(f)$, $f(\alpha)$ is a compact open set witnessing that $\alpha \in \text{Bad}(p, D)$. Since $\text{Bad}(p, D)$ is uncountable, f is a promise and thus $r = (x_p, \Phi_p \cup \{f\})$ is a condition in P_X . Because $[r] \setminus [p] = \emptyset$, r is an extension of p . Now D is dense in P_X so we can find a $q \in D$ that extends r . The definition of extension in P_X implies that $Y = \{\alpha \in \text{Bad}(p, D) : [q] \setminus [r] \subseteq f(\alpha)\}$ is uncountable. Since $[q] \setminus [r] = [q] \setminus [p]$, we know that if $\alpha \in Y$, then $[q] \setminus [p] \subseteq f(\alpha)$, contradicting the fact that $f(\alpha)$ witnesses that $\alpha \in \text{Bad}(p, D)$. \square

Lemma 4.5. *Fix a countable $N \prec H(\lambda)$ with X and P_X in N , and let U be an open subset of X with $U \cap N$ unbounded in $\delta = N \cap \omega_1$. Given a condition $p \in N \cap P_X$ and a dense open subset D of P_X that is in N , there is an extension q of p such that $q \in N \cap D$ and $[q] \setminus [p] \subseteq U$.*

Proof. All parameters involved in the definition of $\text{Bad}(p, D)$ are members of N , so $\text{Bad}(p, D) \in N$ as well. Since $\text{Bad}(p, D)$ is countable, we know it must be bounded below δ , and since $U \cap \delta$ is cofinal in δ , we can find a $\beta < \delta$ such that $\beta \in U \setminus \text{Bad}(p, D)$. By our comments above, we can find a compact open neighborhood V of β such that $V \subseteq \beta + 1 \cap U$ and $V \in N$. Since $\beta \notin \text{Bad}(p, D)$, in N we can find an extension q of p such that $q \in D$ and $[q] \setminus [p] \subseteq V \subseteq U$ as promised. \square

We are almost in a position to show that P_X is totally proper—given a relevant countable model N and a condition $p \in N \cap P_X$, we can fix some compact open neighborhood U of $\delta = N \cap \omega_1$ and repeatedly apply the above lemma to build a decreasing sequence $p = p_0 \geq p_1 \geq \dots$ of conditions in $N \cap P_X$ such that $[p_n] \setminus [p] \subseteq U$ for each n and if $D \in N$ is dense open in P_X , then some p_n is in D . The set $\bigcup_{n \in \omega} [p_n]$ is contained in U modulo a compact set, and hence it will have compact closure. This is not quite enough to ensure that the sequence of p_n 's has a lower bound — somehow we need to make sure that if f is a promise appearing in some Φ_{p_n} then

$$(4.3) \quad \bigcap_{m > n} Y(f, p_m, p_n) \text{ is uncountable}$$

because the lower bound has to keep all promises that have arisen along the way. We need another definition.

Definition 4.6. If f is a promise, define

$$(4.4) \quad H_f = \{\alpha < \omega_1 : \text{there is a set } V_\alpha \text{ that is open, cofinally convergent, and cofinal in } \alpha \text{ with compact closure such that the set } K_{\alpha, f} = \{\beta \in \text{dom}(f) : \text{Top}(V_\alpha) \in f(\beta)\} \text{ is countable}\}.$$

Lemma 4.7. *For any promise f , the set H_f is countable.*

Proof. Suppose this is not the case. Since $\alpha \leq \text{Top}(V_\alpha)$ for each $\alpha \in H_f$, the set $Y = \{\text{Top}(V_\alpha) : \alpha \in H_f\}$ is uncountable, and hence $\text{cl } Y$ is co-countable. Choose ξ large enough so that $\omega_1 \setminus \xi \subseteq \text{cl } Y = \text{cl}(Y \cap \xi)$, and choose a $\gamma \in \text{dom}(f)$ such that $\gamma > \xi$ and $\sup K_{\zeta, f} < \gamma$ for all $\zeta \in H_f \cap \gamma$. But then $Y \cap \xi \cap f(\gamma) = \emptyset$, contradicting the density of $Y \cap \xi$ in $\text{cl } Y$. \square

Lemma 4.8. *Let $N \prec H(\lambda)$ be countable with X and P_X in N . Let U be a cofinally convergent open set, cofinal in $\delta = N \cap \omega_1$, and let $f \in N$ be a promise. Then there*

is a compact open V_f bounded below δ such that $\{\alpha \in \text{dom}(f) : U \setminus f(\alpha) = V_f\}$ is uncountable.

Proof. Since H_f is countable and $H_f \in N$, we know that $\delta \notin H_f$ and thus

$$(4.5) \quad K_{\delta,f} = \{\beta \in \text{dom}(f) : \text{Top}(U) \in f(\beta)\}$$

is uncountable. Now U is cofinally convergent so by Lemma 4.8, $\text{Top}(U) \in f(\beta)$ if and only if there is a compact open W bounded below δ such that $U \setminus f(\beta) = W$. Since U has only countably many compact open subsets, there is a single V_f such that $\{\alpha \in \text{dom}(f) : U \setminus f(\alpha) = V_f\}$ is uncountable. \square

Lemma 4.9. *Given $N \prec H(\lambda)$ countable with X and P_X in N , $p \in N \cap P_X$, and U cofinally convergent, compact open and cofinal in $\delta = N \cap \omega_1$, there is an extension q of p such that q is totally (N, P_X) -generic and $[q] \setminus [p] \subseteq U$.*

Proof. Fix an enumeration $\{D_n : n \in \omega\}$ of the dense open subsets of P_X that are members of N . By iterating Lemma 4.5, we can certainly construct a decreasing sequence of conditions $\{p_n : n \in \omega\}$ starting with $p_0 = p$ such that $p_{n+1} \in N \cap D_n$ and $[p_{n+1}] \setminus [p_n] \subseteq U$. Let $x = \bigcup_{n \in \omega} x_{p_n}$ and $[x] = x^{-1}(\{1\})$. Since $[x] \setminus [p] \subseteq U$, we know that $[x]$ has compact closure. If the sequence $\{p_n : n \in \omega\}$ is said to have a lower bound, then for each $n \in \omega$ and each promise $f \in \Phi_{p_n}$, we need for there to be uncountably many ξ in $\text{dom}(f)$ with $[x] \setminus [p_n] \subseteq f(\xi)$.

To ensure this, we make sure that every f that appears in some Φ_{p_n} gets handled in the following manner:

Assume we have defined p_n and we are given an $f \in \Phi_{p_i}$ for some $i \leq n$ that must be handled at this stage. First, we know that the set

$$Y = \{\xi \in \text{dom}(f) : [p_n] \setminus [p_i] \subseteq f(\xi)\}$$

is uncountable and $f \upharpoonright Y \in \Phi_{p_n}$. Since $f \upharpoonright Y$ is in N , we know by Lemma 4.8 that there is a compact open V bounded below δ such that the set $Z = \{\xi \in Y : U \setminus f(\xi) = V\}$ is uncountable. Let s be an extension of p_n with $V \subseteq \text{dom}(x_s)$ defined by setting $x_s(\xi) = 0$ on any ξ added to $\text{dom}(x_{p_n})$. Since V is bounded below δ , we can find such an s in N . We can apply Lemma 4.5 to get a $p_{n+1} \leq s$ such that $p_{n+1} \in N \cap D_n$ and $[p_{n+1}] \setminus [s] = [p_{n+1}] \setminus [p_n] \subseteq U$. Notice that if r is any extension of p_{n+1} that satisfies $[r] \setminus [p_{n+1}] \subseteq U$, then $[r] \setminus [p_{n+1}] \subseteq f(\xi)$ for all $\xi \in Z$.

Since each condition involves only countably many promises, we can construct $\{p_n : n \in \omega\}$ so that

1. $p_0 = p$, $p_{n+1} \leq p_n$, $p_{n+1} \in N \cap D_n$;
2. for each n and $f \in \Phi_{p_n}$, there is some later stage n_f such that f is handled at stage n_f in the manner described above.

Once this is accomplished, let $x = \bigcup_{n \in \omega} x_{p_n}$ and $[x] = x^{-1}(\{1\})$. We claim that if $f \in \Phi_{p_n}$ for some n , then $Y_f = \{\xi \in \text{dom}(f) : [x] \setminus [p_n] \subseteq f(\xi)\}$ is uncountable. To see this, fix $m \geq n$ such that f was handled when we defined p_{m+1} . Let

$$(4.6) \quad Y = \{\xi \in \text{dom}(f) : [p_m] \setminus [p_n] \subseteq f(\xi)\}.$$

We fixed V such that $Z = \{\xi \in Y : U \setminus f(\xi) = V\}$ is uncountable, and by our comments above, we know that for all $\xi \in Z$ and $i > m$ that $[p_i] \setminus [p_{m+1}] \subseteq f(\xi)$,

so $Z \subseteq Y_f$. Thus if we define q by

$$(4.7) \quad q = (x, \bigcup_{n \in \omega} \Phi_{p_n} \cup \{f \restriction Y_f : f \in \bigcup_{n \in \omega} \Phi_{p_n}\}),$$

then q is a lower bound for the sequence $\{p_n : n \in \omega\}$ in P_X . \square

Theorem 2 of the Introduction now follows as a trivial corollary of Lemma 4.9. A similar proof shows that P_X is in fact $< \omega_1$ -proper. Since this will be needed in the sequel, we present this now.

Proposition 4.10. *If X is an Ostaszewski space, the forcing P_X defined above is $< \omega_1$ -proper.*

Proof. We prove by induction on $\alpha < \omega_1$ that for each countable α -tower $\mathfrak{N} = \langle N_\xi : \xi \leq \alpha \rangle$ with $P_X \in N_0$, any $p \in N_0 \cap P_X$, and any open U open with compact closure and cofinal in $\delta_\alpha = N_\alpha \cap \omega_1$ that there is a $q \leq p$ with $[q] \setminus [p] \subseteq U$ that is totally (N_ξ, P_X) -generic for each $\xi \leq \alpha$. The case where α is a successor ordinal follows from the induction hypothesis by a simple application of Lemma 4.9, so we assume that α is a limit ordinal.

Fix an increasing sequence $\{\alpha_n : n \in \omega\}$ cofinal in α . We construct a decreasing sequence $\{p_n : n \in \omega\}$ satisfying that

1. $p_0 = p$, $p_{n+1} \in N_{\alpha_{n+1}}$ is totally (N_ξ, P_X) -generic for $\xi \leq \alpha_n$,
2. $[p_{n+1}] \setminus [p_n] \subseteq U$,
3. each promise f appearing in some Φ_{p_n} gets handled at some stage in the manner described below.

Assume we have p_n satisfying 1 and 2 above, and we are given a promise f from some Φ_{p_i} where $i \leq n$. We know that $Y(f, p_i, p_n)$ is uncountable, and $g = f \restriction Y(f, p_i, p_n) \in \Phi_{p_n}$. By Proposition 2.12, $U \cap N_{\alpha_{n+1}}$ is cofinal in $\delta_{\alpha_{n+1}} = N_{\alpha_{n+1}} \cap \omega_1$, so there is a compact open V_g bounded below $\delta_{\alpha_{n+1}}$ such that

$$(4.8) \quad Z_g = \{\xi \in \text{dom}(g) : U \setminus g(\xi) = V_g\}$$

is uncountable.

In $N_{\alpha_{n+1}}$, extend p_n to a condition s satisfying $[p_n] = [s]$ and $V_g \subseteq \text{dom}(x_s)$, and then apply our induction hypotheses to s , U , and the tower $\langle N_\xi : \alpha_n < \xi \leq \alpha_{n+1} \rangle$ to get $p_{n+1} \in N_{\alpha_{n+1}+1}$ as needed. Just as in the proof of Lemma 4.9, if $r \leq p_{n+1}$ satisfies $[r] \setminus [p_{n+1}] \subseteq U$, then $Z_g \subseteq Y(g, r, p_{n+1})$, and an argument almost identical to the one given there shows us that $\{p_n : n \in \omega\}$ has a lower bound q which is (N_ξ, P_X) -generic for $\xi \leq \alpha$ and $[q] \setminus [p] \subseteq U$. \square

5. TWO STEP ITERATIONS

Now that we know we can destroy an Ostaszewski space without adding new reals, the next problem is to show that an iteration of such forcings does not add reals.

To start our discussion of this problem, assume that P is a totally proper notion of forcing and \dot{Q} is a P -name for a totally proper notion of forcing. Fix a countable elementary submodel N of some $H(\lambda)$ that contains both P and \dot{Q} . The question we need to address is: If r is totally (N, P) -generic, can we find a P -name \dot{s} so that $r * \dot{s}$ is totally $(N, P * \dot{Q})$ -generic?

The obvious thing to try is to choose \dot{s} such that

$$(5.1) \quad r \Vdash \dot{s} \text{ is totally } (N[\dot{G}_P], \dot{Q})\text{-generic.}$$

If $D \in N$ is dense and open in $P * \dot{Q}$, then certainly

$$(5.2) \quad r * \dot{s} \Vdash \text{there is a } t \in N \cap D \text{ such that } r * \dot{s} \leq t$$

but $r * \dot{s}$ need not be an extension of any particular $t \in N \cap D$, and hence not totally $(N, P * \dot{Q})$ -generic.

What we need is a sequence $\{\dot{q}_n : n \in \omega\}$ of P -names from N (which is a stronger requirement than $r \Vdash \dot{q}_n \in N[\dot{G}]$) so that

$$(5.3) \quad r \Vdash \{\dot{q}_n : n \in \omega\} \text{ generates a member of } \text{Gen}^+(N[\dot{G}], \dot{Q}).$$

If \dot{s} is forced by r to extend each \dot{q}_n , then it is not hard to see that $r * \dot{s}$ is totally $(N, P * \dot{Q})$ -generic. Thus we need to address the question: What does it take for r to see such a sequence of P -names?

In the situation of interest to us, namely when there is a P -name \dot{X} for an Ostaszewski space and \dot{Q} is a name for the forcing $P_{\dot{X}}$ introduced in the previous section, we can get a reasonable condition on r that guarantees the existence of a suitable \dot{s} .

We will actually need something a bit stronger than this in the proof of our iteration theorem, but in order to see where we are headed, we need another definition.

Definition 5.1. Assume that

1. P is totally proper, and $\Vdash_P \dot{Q}$ is totally proper,
2. $N \prec H(\lambda)$ is countable with $\{P, \dot{Q}\} \in N$,
3. $\bar{G} \in \text{Gen}(N, P)$,

Then $\bar{H} \in \text{Gen}(N[\bar{G}], \dot{Q})$ means

4. \bar{H} consists of P -names from N ,
5. $\{\dot{s}[\bar{G}] : \dot{s} \in \bar{H}\}$ is an $N[\bar{G}]$ -generic subset of $\dot{Q}[\bar{G}]$.

Similarly, if $\dot{q} \in N$ is a P -name for an element of \dot{Q} , then we let

$$\text{Gen}(N[\bar{G}], \dot{Q}, \dot{q}) = \{\bar{H} \in \text{Gen}(N[\bar{G}], \dot{Q}) : \dot{q} \in \bar{H}\}.$$

Now our goal is to prove the following theorem which will be crucial in our proof that our forcing can be iterated without adding reals.

Theorem 3. Assume

1. P is totally proper,
2. \dot{X} is a P -name for an Ostaszewski space,
3. \dot{Q} is a P -name for $P_{\dot{X}}$, the notion of forcing that destroys \dot{X} ,
4. $N_0 \in N_1 \in N_2$ are countable elementary submodels of $H(\lambda)$,
5. $\{P, \dot{X}, \dot{Q}\} \in N_0$,
6. $\bar{G} \in \text{Gen}^+(N_0, P) \cap N_1$,
7. $\dot{q} \in N$ is a P -name for a condition in \dot{Q} .

Then we can find $\bar{H} \in \text{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ so that whenever r is a lower bound for \bar{G} that is (N_i, P) -generic for $i = 1, 2$, we have

$$r \Vdash \bar{H} \text{ has a lower bound in } \dot{Q}.$$

Notice the order of the quantifiers here — we build a single \bar{H} that works for all relevant r . The proof of this will appear later in the section. Even though it may seem hopeless to build such an \bar{H} that takes into account all of the relevant r , the key is Lemma 3.8. All r of concern to us have the same general idea of what $N_1 \cap \dot{G}$ might be — they all agree it must lie in $N_2 \cap \text{Gen}(N_1, P)$.

We add to the proof of Theorem 3 by considering our situation in detail. Assume that P is a totally proper notion of forcing and N is a countable elementary submodel of some $H(\lambda)$ such that $P \in N$. Further assume that $\dot{X} \in N$ is a name forced by every condition in P to be an Ostaszewski space with underlying set ω_1 and initial segments open.

If \bar{G} is an element of $\text{Gen}^+(N_0, P)$ and $\alpha \in N$ is a countable ordinal, then there is a P -name $\dot{\tau}_\alpha$ for a countable family of subsets of $\alpha + 1$ such that every condition in P forces $\dot{\tau}_\alpha$ is a base for the topology of \dot{X} at α . Because P is totally proper, $\dot{\tau}_\alpha$ is forced to be an element of the ground model, and thus some condition in \bar{G} decides a specific value for $\dot{\tau}_\alpha$.

If we let $\delta = N \cap \omega_1$, then by decoding all the names $\dot{\tau}_\alpha$ using \bar{G} we get a first countable, locally countable topology on δ , and any lower bound for \bar{G} forces that this is in fact the topology on δ viewed as a subspace of \dot{X} . We let X_δ denote this topology that we get from \bar{G} .

Let us step, for a moment, into the model $N[\bar{G}]$ defined as in Definition 3.4. As far as $N[\bar{G}]$ is concerned $\dot{X}[\bar{G}]$ is an Ostaszewski space, and in fact $\dot{X}[\bar{G}] = X_\delta$. This follows because in the generic extension, the Mostowski collapse of $N[G]$ (where G is any generic subset of P containing a lower bound for \bar{G}) is the identity map when restricted to the hereditarily countable sets.

We now have the background necessary to prove Theorem 3, but it is instructive at this point to consider first a simplified version of our goal.

Definition 5.2. In the situation of Theorem 3, we say $r \in P$ is generic enough if r is a lower bound for \bar{G} that is (N_i, P) -generic for $i = 1, 2$.

Proposition 5.3. In the situation of Theorem 3, we can find a set V open in X_δ such that any generic enough r will force V to be cofinally convergent in δ .

Proof. First, we note that the space X_δ is in N_1 because it is definable from \bar{G} and P -names from N_0 . Let $\{\bar{H}_n : n \in \omega\}$ list all members of $\text{Gen}^+(N_1, P) \cap N_2$ that extend \bar{G} ; since N_1 contains both \bar{G} and a lower bound for \bar{G} , this set is non-empty. By Lemma 3.5, for each n

$$(5.4) \quad N_1[\bar{H}_n] \models \dot{X}[\bar{H}_n] \text{ is an Ostaszewski space and } X_\delta \text{ is a subspace of } \dot{X}[\bar{H}_n].$$

Let A_n be such that

$$(5.5) \quad N_1[\bar{H}_n] \models A_n = \{U \subseteq \delta : U \text{ is open and cofinally convergent in } \delta\}.$$

Notice that each $U \in A_n$ is open in X_δ and since P is totally proper, $A_n \cap N_1[\bar{H}_n] \subseteq N_1$. Most importantly, Corollary 2.15 implies that whenever $U \in N_1$ is cofinal in δ and open in X_δ , there is a $V \subseteq U$ such that $V \in N_1 \cap A_n$. This is the only place where the countable compactness of Ostaszewski spaces is used in our proof.

Each $\dot{X}[\bar{H}_n]$ should be thought of as a guess at the topology on $\dot{X} \cap \delta_1$ (where $\delta_1 = N_1 \cap \omega_1$), and by Lemma 3.8, we know that any generic enough r will force that one of these guesses is the correct topology in the generic extension. We construct an open $W \subseteq X_\delta$, cofinal in δ , such that W will be cofinally convergent no matter which of these guesses turns out to be the right one.

To do this, fix an increasing ω -sequence $\{\alpha_n : n \in \omega\}$ cofinal in δ . We define by recursion sets U_n and W_n such that

1. $U_0 = \delta$,
2. $U_{n+1} \in N_1 \cap A_n$,

3. $U_{n+1} \subseteq U_n \setminus (W_0 \cup \dots \cup W_n)$,
4. $W_n \in N_0[\bar{G}]$, W_n compact open in X_δ , $W_n \subseteq U_n$, $\max(W_n) \geq \alpha_n$.

We can always find U_{n+1} because $U_n \setminus (W_0 \cup \dots \cup W_n) \in N_1$, so we can apply Corollary 2.15 in $N_1[\bar{H}_n]$.

Let $W = \bigcup_{n \in \omega} W_n$. Condition 4 ensures that W is an open subset of X_δ that is cofinal in δ .

We are still stuck with the task of showing that W is forced to be cofinally convergent by any generic enough r . Fix such an r , and let G be any generic subset of P that contains r . By Lemma 3.8, we know there is some n such that $N_1 \cap G = \bar{H}_n$, and so U_{n+1} is cofinally convergent in δ in $V[G]$. Since $W \setminus (W_0 \cup \dots \cup W_n) \subseteq U_{n+1}$, W is cofinally convergent in δ as well. Since G was an arbitrary generic set containing r , we have then that $r \Vdash W$ is cofinally convergent in δ . \square

Now in order to prove Theorem 3, we use the same strategy — if we know the correct topology on X_δ and we have countably many guesses at the correct topology on $\dot{X} \cap \delta_1$, one of which is guaranteed to be correct, we can build the desired sequence by diagonalization. Notice also that a single diagonalization argument will take care of all generic enough r 's just as in the proof above. We've already seen how to guarantee the union of the first components of the \dot{q}_n 's has compact closure, so the only added complication is all the promises that arise along the way.

Proof of Theorem 3. Let $\{\dot{D}_n : n \in \omega\}$ list all P -names from N_0 of dense open subsets of \dot{Q} , and let $\{\bar{H}_n : n \in \omega\}$ enumerate those members of $\text{Gen}^+(N_1, P) \cap N_2$ that extend \bar{G} , and let A_n be defined as in (5.5). The construction of $\{\dot{q}_n : n \in \omega\}$ is essentially a combination of Lemma 4.9 and Proposition 5.3.

Before commencing with the construction, we make a convention that when dealing with P -names $\dot{\sigma}$ from N_0 , we identify $\dot{\sigma}$ and $\dot{\sigma}[\bar{G}]$. Thus when we say $\dot{q}_{n+1} \in \dot{D}_n$, we mean $N_0[\bar{G}] \models \dot{q}_{n+1}[\bar{G}] \in \dot{D}_n[\bar{G}]$, when we write $[\dot{q}_n]$, we mean $[\dot{q}_n[\bar{G}]]$.

We define for each $n \in \omega$ a set U_n and a P -name $\dot{q}_n \in N_0$ such that

1. $U_0 = \delta$, $\dot{q}_0 \in N_0$ any P -name for a condition in \dot{Q} ,
2. $U_{n+1} \in N_1 \cap A_n$,
3. $U_{n+1} \subseteq U_n \setminus \text{dom}(x_{\dot{q}_n})$,
4. $\dot{q}_{n+1} \in \dot{D}_n$,
5. $[\dot{q}_{n+1}] \setminus [\dot{q}_n] \subseteq U_{n+1}$,
6. each promise \dot{f} appearing in some \dot{q}_n gets taken care of with respect to each \bar{H}_m in a sense to be made precise in a moment.

At a stage $n+1$, we can choose U_{n+1} satisfying 2 and 3 precisely as in the proof of the preceding proposition. Since U_{n+1} is open and cofinal in δ , we can apply Lemma 4.5 in $N_1[\bar{H}_n]$ to get \dot{q}_{n+1} satisfying 4 and 5.

For condition 6, if \dot{f} is a promise appearing in some \dot{q}_n where $n > m$, then because \bar{H}_m contains a lower bound for \bar{G} ,

$$(5.6) \quad N_1[\bar{H}_m] \models \dot{f}[\bar{H}_m] \text{ is a promise in } \dot{q}_n[\bar{H}_m].$$

In the model $N_1[\bar{H}_m]$, after the set U_{m+1} has been defined, we may apply Lemma 4.8 to get a compact open $V(\dot{f}, m)$ bounded below δ such that

$$(5.7) \quad N_1[\bar{H}_m] \models Z(\dot{f}, m) = \{\alpha \in \text{dom}(\dot{f}[\bar{H}_m]) : \\ U_{m+1} \setminus \left(\dot{f}[\bar{H}_m](\alpha) \right) = V(\dot{f}, m)\} \text{ is uncountable.}$$

Note that Lemma 4.8 applies because U_{m+1} is cofinally convergent in $N_1[\bar{H}_m]$ and $\dot{f}[\bar{H}_m] \in N_0[\bar{H}_m] \in N_1[\bar{H}_m]$.

When we say “ \dot{f} (a promise from \dot{q}_n) gets handled with respect to \bar{H}_m at stage $i > m$ ”, we mean when it comes time to define \dot{q}_{i+1} we do the following:

In $N_0[\dot{G}]$, let

$$(5.8) \quad \dot{Y} = Y(\dot{f}, \dot{q}_n, \dot{q}_i) = \{\xi \in \text{dom}(\dot{f}) : [\dot{q}_i] \setminus [\dot{q}_n] \subseteq \dot{f}(\xi)\}.$$

By definition of the extension in \dot{Q} , we know that \dot{Y} is uncountable and $\dot{f} \upharpoonright \dot{Y}$ is a promise in the condition \dot{q}_i . Let \dot{s} be an extension of \dot{q}_i with $[\dot{s}] = [\dot{q}_i]$ and $V(\dot{f} \upharpoonright \dot{Y}, m)$ contained in the domain of $\dot{x}_{\dot{q}_i}$, and then let \dot{q}_{i+1} be an extension of \dot{s} satisfying 4 and 5. Note that $V(\dot{f}, m)$ was defined using the “privileged information” obtained from \bar{H}_m , but it is still a P -name from N_0 .

Since only countably many such \dot{f} appear in our construction of $\{\dot{q}_n : n \in \omega\}$, we can arrange that each \dot{f} gets handled with respect to each \bar{H}_m at some stage $i > m$.

Now let r be generic enough, and let G be any generic subset of P that contains r . By Lemma 3.8, there is an n such that $N_1 \cap G = \bar{H}_n$. Thus U_{n+1} is cofinally convergent and $\bigcup_{n \in \omega} [\dot{q}_n]$ will have compact closure. If f is a promise appearing in some $\dot{q}_n[G]$, then f is of the form $\dot{f}[G]$ for some $\dot{f} \in N_0$ (because $\dot{q}_n \in N_0$ contains only countably many promises), and so there is some stage $i > n$ at which \dot{f} was handled with respect to \bar{H}_n . Just as in the proof of Lemma 4.9, whenever \dot{s} is any extension of \dot{q}_{n+1} in N_0 that satisfies $[\dot{s}] \setminus [\dot{q}_{n+1}] \subseteq U_{n+1}$, we have

$$(5.9) \quad Y(\dot{f}, \dot{s}, \dot{q}_{n+1}) \supseteq Z(\dot{f}, n).$$

Since our sequence $\{U_i : i \in \omega\}$ is decreasing, we know that for $m \geq n+1$,

$$(5.10) \quad [\dot{q}_m] \setminus [\dot{q}_{n+1}] \subseteq U_{n+1}.$$

Thus if we let $x = \bigcup_{i \in \omega} x_{\dot{q}_i}$ and $[x] = x^{-1}(\{1\})$, then

$$(5.11) \quad Y_f = \{\xi \in \text{dom}(\dot{f}[G]) : [x] \setminus [\dot{q}_n[G]] \subseteq \dot{f}[G](\xi)\}$$

contains $Z(\dot{f}, n)$, and hence is uncountable.

Thus if we define Φ in $V[G]$ as the collection of all promises appearing in some $\dot{q}_i[G]$ together with $f \upharpoonright Y_f$ for each promise f that appears in some $\dot{q}_i[G]$, then $q = (x, \Phi)$ is a lower bound for $\{\dot{q}_i : i \in \omega\}$. Since G was an arbitrary generic subset of P that contains r , we have

$$(5.12) \quad r \Vdash \{\dot{q}_n : n \in \omega\} \text{ generates a member of } \text{Gen}^+(N_0[\dot{G}], \dot{Q}).$$

Thus from the sequence $\{\dot{q}_n : n \in \omega\}$ we get the desired \bar{H} . □

6. 2-COMPLETENESS AND AN ITERATION THEOREM

The property of P_X isolated in Theorem 3 is what allows us to prove that a countable support iteration of such forcings does not add reals. This is the motivation behind the following definition.

Definition 6.1. Let P be totally proper and let \dot{Q} be a P -name for a totally proper notion of forcing. We say \dot{Q} is 2-complete for P if whenever

1. $N_0 \in N_1 \in N_2$ are countable elementary submodels of $H(\lambda)$ with $P, \dot{Q} \in N_0$,
2. $\bar{G} \in \text{Gen}^+(N_0, P) \cap N_1$,
3. $\dot{q} \in N_0$ is a P -name for a condition in \dot{Q} ,

there is an $\bar{H} \in \text{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ so that whenever r is a lower bound for \bar{G} that is (N_i, P) -generic for $i = 1, 2$, we have

$$r \Vdash \bar{H} \text{ has a lower bound in } \dot{Q}.$$

Thus Theorem 3 tells us that if P is totally proper and \dot{X} is a P -name for an Ostaszewski space, then $P_{\dot{X}}$ is 2-complete for P .

The proof of the following theorem is, in some sense, isomorphic to the proof of Theorem 7.1 of Chapter V in [5]. In place of Shelah's \mathbb{D} -completeness, a condition on individual notions of forcing, we use 2-completeness, a condition on iterations. The actual differences between our proof and Shelah's are only minor technical details — the underlying idea is unchanged.

Theorem 4. Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$ be a countable support iteration such that

- $\Vdash_\alpha \dot{Q}_\alpha$ is $< \omega_1$ -proper,
- \dot{Q}_α is 2-complete for P_α .

Then P_κ is totally proper.

Proof. The theorem is easy if κ is finite, so we assume that κ is a limit ordinal. This reduction is purely for typographical convenience, since otherwise we would have to replace the κ^* defined below by $2\kappa^*$.

Let $N \prec H(\lambda)$ be countable with $\mathbb{P} \in N$, and let $p \in P_\kappa \cap N$ be arbitrary. We must produce a totally (N, P_κ) -generic condition $q \leq p$. For $\theta \in N \cap (\kappa + 1)$ let $\theta^* = \text{otp}(N \cap \theta)$. Also fix an elementary tower $\mathfrak{N} = \langle N_\xi : \xi \leq \kappa^* \rangle$ such that $N = N_0$. In order to facilitate the statement of the following claim, we let N_{κ^*+1} stand for V .

Claim 6.2. Given $\alpha < \beta$ in $N_0 \cap (\kappa + 1)$ and $G \in \text{Gen}^+(N_0, P_\alpha, p \restriction \alpha) \cap N_{2\alpha^*+1}$, there is a $G^\dagger \in \text{Gen}(N_0, P_\beta, p) \cap N_{2\beta^*+1}$ such that whenever $r \in P_\alpha$ is a lower bound for G that is (N_ξ, P_α) -generic for $2\alpha^* < \xi \leq \kappa^*$, there is an $r^\dagger \in P_\beta$ such that

- (a) r^\dagger is a lower bound for G^\dagger ,
- (b) $r^\dagger \restriction \alpha = r$,
- (c) r^\dagger is (N_ξ, P_α) -generic for any ξ with $2\beta^* < \xi \leq \kappa^*$.

Proof of Claim 6.2. The proof is by induction on $\beta \in N_0 \cap (\kappa + 1) \setminus \{0\}$, for all $\alpha \in N_0 \cap \beta$.

Case 1: $\alpha = 0, \beta = 1$. Recall that our convention is that P_0 is the trivial poset. The claim then reduces to the fact that \dot{Q}_0 is totally proper and $< \omega_1$ -proper.

Case 2: $\beta = \beta_0 + 1$. Note that $\beta_0 \in N_0 \cap \beta$ as well. If $\alpha < \beta_0$, an application of the induction hypothesis with β_0 in place of β gives us a G_0^\dagger and r_0^\dagger that we can

use together with the following argument for the case $\alpha = \beta_0$. Thus we may as well assume that $\alpha = \beta_0$.

Inside the model $N_{2\beta^*+1}$, we apply the definition of “ \dot{Q}_α is 2-complete for P_α ” with N_0 , $N_{2\alpha^*+1}$, $N_{2\beta^*}$, G , and $p(\alpha)$ in place of the N_0 , N_1 , N_2 , \bar{G} , and \dot{q} there.

This gives us an $\bar{H} \in \text{Gen}(N_0[\dot{G}], \dot{Q}_\alpha, p(\alpha)) \cap N_{2\beta^*+1}$ so that whenever r is a lower bound for G that is (N_ξ, P_α) -generic for $\xi = 2\alpha^* + 1, 2\beta^*$,

$$r \Vdash \bar{H} \text{ has a lower bound in } \dot{Q}_\alpha.$$

Let $\dot{s} \in N_{2\beta^*+1}$ be a P_α -name forced by every such r to be a lower bound for \bar{H} .

Define G^\dagger as $\{u * \dot{v} : u \in G \text{ and } \dot{v} \in \bar{H}\}$. Standard arguments show that $G^\dagger \in \text{Gen}(N_0, P_\beta, p)$, and $G^\dagger \in N_{2\beta^*+1}$.

Now suppose r is any lower bound for G that is (N_ξ, P_α) -generic for $2\alpha^* < \xi \leq \kappa^*$. Then r forces that \dot{s} is a lower bound for \bar{H} , and furthermore, there is \dot{t} such that

$$r \Vdash \dot{t} \leq \dot{s} \text{ is } (N_\xi[\dot{G}], \dot{Q}_\alpha)\text{-generic for } 2\beta^* < \xi \leq \kappa^*,$$

as $\Vdash_\alpha \dot{Q}_\alpha$ is $< \omega_1$ -proper. We finish by defining $r^\dagger = r * \dot{t}$.

Case 3: β a limit ordinal. Work for the moment in the model $N_{2\beta^*+1}$. Fix $\{\alpha_n : n \in \omega\}$ strictly increasing and cofinal in $N_0 \cap \beta$, with $\alpha_0 = \alpha$. Let $\{D_n : n \in \omega\}$ list those dense open subsets of P_β that are members of N_0 .

We will define $\{p_n : n \in \omega\}$ and $\{G_n : n \in \omega\}$ that satisfy

1. $p_0 = p$, $G_0 = G$;
2. $p_{n+1} \in N_0 \cap D_n$;
3. $p_{n+1} \leq p_n$;
4. $p_n \restriction \alpha_n \in G_n$;
5. $G_n \in \text{Gen}(N_0, P_{\alpha_n}, p \restriction \alpha_n) \cap N_{2\alpha_n^*+1}$;
6. whenever $r \in P_{\alpha_n}$ is a lower bound for G_n that is (N_ξ, P_{α_n}) -generic for $2\alpha_n^* < \xi \leq \kappa^*$, there is an $r' \in P_{\alpha_{n+1}}$ such that
 - r' is a lower bound for G_{n+1} ,
 - $r' \restriction \alpha_n = r$,
 - r' is $(N_\xi, P_{\alpha_{n+1}})$ -generic for any ξ with $2\alpha_{n+1}^* < \xi \leq \kappa^*$.

Given p_n and G_n , note that $E = \{t \restriction \alpha_n : t \in D_n, t \leq p_n\}$ is dense below $p_n \restriction \alpha_n$ and an element of N_0 . Since $p_n \restriction \alpha_n \in G_n$, we can find $t \in E \cap G_n$. Applying the definition of E inside of N_0 gives us a $p_{n+1} \in N_0 \cap D_n$ with $p_{n+1} \leq p_n$ and $p_{n+1} \restriction \alpha_n \in G_n$.

The required G_{n+1} is obtained by applying our main induction hypothesis with α_n , α_{n+1} , $p_{n+1} \restriction \alpha_{n+1}$, and G_n in place of α , β , p , and G .

Once we have the sequences $\{p_n : n \in \omega\}$ and $\{G_n : n \in \omega\}$, define

$$G^\dagger = \{t \in N_0 \cap P_\beta : \exists n[t \geq p_n]\}.$$

Our construction guarantees that $G^\dagger \in \text{Gen}(N_0, P_\beta, p)$, and since we did the construction inside of $N_{2\beta^*+1}$, we know G^\dagger is in this model as well.

Subclaim 1. G^\dagger has a lower bound.

To see this, let r be a lower bound for G that is (N_ξ, P_α) -generic for $2\alpha^* < \xi \leq \kappa^*$. The properties of the sequence $\{G_n : n \in \omega\}$ allow us to build a sequence $\{r_n : n \in \omega\}$ satisfying

- $r_0 = r$,

- r_n is a lower bound for G_n in P_{α_n} ,
- r_n is (N_ξ, P_{α_n}) -generic for $2\alpha_n^* < \xi \leq \kappa^*$,
- $r_{n+1} \restriction \alpha_n = r_n$.

Finally, we let $r^+ = \bigcup r_n \in P_\beta$.

Now why is r^+ a lower bound for G^\dagger ? Since $r_{n+1} \restriction \alpha_n = r_n$ and $G_{n+1} \restriction P_{\alpha_n}$ is a member of $\text{Gen}(N_0, P_{\alpha_n})$, by Lemma 3.6 we have that $G_{n+1} \restriction P_{\alpha_n} = G_n$. From this, it follows easily by induction that $G_m \restriction P_{\alpha_n} = G_n$ for $m > n$.

For $n \in \omega$, we know that $p_n \restriction \alpha_n \in G_n$. If $m < n$, then $p_m \restriction \alpha_n \geq p_n \restriction \alpha_n$, so $p_m \restriction \alpha_n \in G_n$. If $m > n$, then $p_m \restriction \alpha_n \in G_m \restriction \alpha_n = G_n$. Thus for each n , we have

$$\{p_m \restriction \alpha_n : m \in \omega\} \subseteq G_n.$$

Since $r_n = r^+ \restriction \alpha_n$ is a lower bound for G_n , it follows that r^+ is a lower bound for G^\dagger and the subclaim holds.

To finish the proof of the claim for this case, let $r \in P_\alpha$ be a lower bound for G that is (N_ξ, P_α) -generic for $2\alpha^* < \xi \leq \kappa^*$; we must produce a suitable r^\dagger .

Let $H \subseteq P_\alpha$ be generic with $r \in H$, and step into the extension $V[H]$. Recall that Lemma 3.6 implies $N_0 \cap H = G$.

The proof of the subclaim tells us (in V) that there is a lower bound r^+ for G^\dagger with $r^+ \restriction \alpha = r$. Since $r \in H$, this means (in $V[H]$) that G^\dagger has a lower bound in P_β/H (recall Equation (3.1)), and by elementarity this must hold in the model $N_{2\beta^*+1}[H]$ as well. Thus we can find $s \in P_\beta \cap N_{2\beta^*+1}$ with $s \restriction \alpha \in H$ that is a lower bound for G^\dagger . Since \mathbb{P} is a countable support iteration of $< \omega_1$ -proper notions of forcing, we can find a $t \leq s$ in P_β/H that is $(N_\xi[H], P_\beta/H)$ -generic for $2\beta^* < \xi \leq \kappa^*$ and such that the support of t is a subset of $N_{\kappa^*}[H] \cap On$.

Back in V , we have a P_α -name \dot{t} such that r forces

- $\dot{t} \in P_\beta/\dot{G}_{P_\alpha}$,
- \dot{t} is a lower bound for G^\dagger ,
- \dot{t} is $(N_\xi[\dot{G}_{P_\alpha}], P_\beta/\dot{G}_{P_\alpha})$ -generic for $2\beta^* < \xi \leq \kappa^*$,
- the support of \dot{t} is a subset of $N_{\kappa^*}[\dot{G}_{P_\alpha}] \cap On$.

Since r is (N_{κ^*}, P_α) -generic, the last clause implies r forces that the support of \dot{t} is a subset of $N_{\kappa^*} \cap On$. Thus the condition $r * \dot{t} \in P_\alpha * P_\beta/\dot{G}_{P_\alpha}$ has a canonical translation into a condition $r^\dagger \in P_\beta$ with $r^\dagger \restriction \alpha = r$. The needed properties of r^\dagger then follow immediately from the corresponding properties of $r * \dot{t}$. \square

To finish the proof of Theorem 4, we apply the claim with $\alpha = 0$ and $\beta = \kappa$. The r and G are trivially available, since P_0 is by convention the trivial 1-element poset. This gives us a $G^\dagger \in \text{Gen}(N_0, P_\kappa, p)$ with lower bound r^\dagger , and we are done. \square

Armed with the previous theorem, we are now in position to build a model of $ZFC + CH$ in which there are no Ostaszewski spaces.

Proof of Theorem 1. Starting with a model of CH , construct a countable support iteration $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ so that for each $\alpha < \omega_2$,

$$\Vdash_\alpha \dot{Q} = P_{\dot{X}} \text{ for some Ostaszewski space } \dot{X}.$$

(If at some stage there is no Ostaszewski space, we need iterate no farther.) Assuming that CH holds, P_X is ω_1 -centered — any two conditions with the same first component are compatible — and this is enough to ensure that P_{ω_2} has the ω_2 -c.c. (see Chapter VII of [5]). Since the topology on an Ostaszewski space has size ω_1 , a standard bookkeeping argument allows us to catch our tail and ensure there are

no Ostaszewski spaces in $V[G_{\omega_2}]$. By Theorem 4, we know that CH continues to hold in $V[G_{\omega_2}]$, and so Theorem 1 is established. \square

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